Volatility components, leverage effects, and the return–volatility relations

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This paper investigates the risk-return trade-off by taking into account the model specification problem. Market volatility is modeled to have two components, one due to the diffusion risk and the other due to the jump risk. The model implies Merton’s ICAPM in the absence of leverage effects, whereas the return–volatility relations are determined by interactions between risk premia and leverage effects in the presence of leverage effects. Empirically, I find a robust negative relationship between the expected excess return and the jump volatility and a robust negative relationship between the expected excess return and the unexpected diffusion volatility. The latter provides an indirect evidence of the positive relationship between the expected excess return and the diffusion volatility.

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1. Introduction

The return–volatility relation is a fundamental issue in asset pricing theory. Merton’s (1973, 1980) intertemporal capital asset pricing model (ICAPM) shows that the expected excess return on the market portfolio should be positively related to its conditional volatility. The intuition behind this positive relation is that if volatility is priced by the market, an increase in expected volatility would require a higher return on the portfolio. This is often referred to as the volatility feedback effect (Pindyck, 1984). Whereas the ICAPM and the volatility feedback effect imply a positive relation, the leverage effect, which is firstly discussed by Black (1976) and Christie (1982), indicates that a decrease in stock price causes an increase in the debt-to-equity ratio, which makes the stock riskier and increases its volatility. The investigation in this strand generally finds that volatility responds asymmetrically to negative and positive returns.

Empirical studies on the return–volatility relation have resulted in mixed findings. On the one hand, French et al. (1987), Chou (1988), Campbell and Hentschel (1992), Ghysels et al. (2004), Guo and Whitelaw (2006), and Christensen and Nielsen (2007) have reported the positive and significant relation. On the other hand, others (Campbell, 1987; Breen et al., 1989; Turner et al., 1989; Chou et al., 1992; Glosten et al., 1993; Brandt and Kang, 2004) have documented a negative and/or insignificant relation. Furthermore, the return–volatility relation is very sensitive to the length of the return horizon and the selection of exogenous predictors (Harvey, 2001; Harrison and Zhang, 1999).

Since stock price volatility cannot be directly observed, most of the above studies apply either nonparametric methods or parametric GARCH models to estimating volatility. However, these methods may suffer from the volatility estimate bias since they rarely investigate the model specification problem. Harrison and Zhang (1999) show that GARCH models may be misspecified and could lead to an inaccurate relation. Furthermore, market volatility may consist of different components (Engle and Lee, 1999), which could have different relations with the expected excess return. Simply regressing excess returns on aggregate volatility may not find the exact relations.

In this paper, with reference to the burgeoning literature in continuous-time financial modeling, I closely study the stock price/volatility dynamics and the time-series return–volatility relation. This paper is different from previous studies in the following respects. First, the model specification problem is taken into account. Time-varying volatility and jumps in the market index are well documented. It is also found that jumps arrive at different rates and tend to be clustered (Maheu and McCurdy, 2004; Huang and Wu, 2004; Christoffersen et al., 2009; Chang et al., 2010). Therefore, aggregate market volatility should be contributed
by two different sources: volatility due to the diffusion risk and volatility due to the jump risk.

Second, the paper shows that different components of market volatility behave differently and influence expected returns in different ways. The diffusion volatility is highly persistent and has a small volatility variation, and it plays an important role in governing market volatility in the long-run. On the contrary, the jump volatility is mean-reverting very fast and contributes to market volatility at the short horizon. Even though the time-series investigation of the relationship between expected excess returns and aggregate market volatility has been considerably conducted, studies on volatility components have attracted less attention. In particular, the relationship between the expected excess return and the jump volatility has not been fully examined to date.

Third, the paper sheds new lights on the risk-return trade-off. The model implies Merton’s ICAPM in the absence of leverage effects, whereas the return–volatility relations are determined by interactions between risk premia and leverage effects in the presence of leverage effects. The model nests French et al. (1987) indirect regression as a special case, and it provides a theoretical explanation for mixed empirical findings on the return–volatility relation obtained by previous studies.

Fourth, the paper proposes an efficient Bayesian method to extract market volatility components. Bayesian estimation with MCMC methods is particularly suitable to the continuous-time financial models (Johannes and Polson, 2009). Eraker et al. (2003) use Bayesian methods to estimate the jump-diffusion stochastic volatility models. Li et al. (2008) estimate one-volatility-factor Lévy models using a similar approach. In this paper, instead of using the traditional Metropolis methods, I propose to apply the more efficient slice sampling method recently developed by Neal (2003). Slice sampler is an adaptive algorithm and is more convenient in drawing samples from posterior distributions of latent states.

Different from most of previous studies that use monthly data, this paper uses a long time series of S&P 500 index data in daily frequency ranging from January 4, 1960 to September 30, 2009 (in total, 12,522 observations). Daily frequency data are very important for identifying the jump-related parameters and the jump volatility component. Empirically, I find that the effect of the diffusion volatility can last nearly half year, whereas the effect of the jump volatility is very short-lived. It is also found that the market regards the 1987’s crash as huge jumps, but the recent financial crisis as high diffusion volatility.

The empirical study finds a negative return–jump volatility relation, which is statistically significant and is very robust across different time periods. For example, for the whole period, the relation is about −0.016; for the pre-1987’s market crash period, this relation is about −0.011; for the post-1987’s market crash and before-Millennium period, it is around −0.15; and for the after-Millennium period, it is slightly small and becomes −0.026. The relation is highly statistically significant for each of these periods. However, it is difficult to identify the relationship between the expected excess return and the diffusion volatility. For example, for the whole period, the return–diffusion volatility relation is insignificantly negative; it is insignificantly positive for the pre-1987’s market crash period; but it becomes significantly positive for the post-1987’s market crash and before-Millennium period.

The French et al. (1987) regression indicates existence of the indirect evidence: the excess return has a negative relationship with the unexpected diffusion volatility, and this negative relation provides an indirect evidence of the positive relationship between the expected excess return and the diffusion volatility. Different relations between the expected excess return and the diffusion/jump volatility can be explained by their behaviors in governing aggregate market volatility. If the current diffusion volatility is higher than the predicted, the predicted diffusion volatility will be revised upward for all future periods because the diffusion volatility is very persistent and has a small volatility variation. If the excess return is positively related to the conditional diffusion volatility, the discount rate for future cash flows will increase. The higher discount rate reduces both their present value and the current stock price if the cash flows are not affected. However, since the jump volatility is mean-reverting very fast, it will be revised downward in the near future if the predicted jump volatility is larger than the current one, and thus a negative return–jump volatility relation is implied.

The rest of paper is organized as follows. Section 2 discusses the model specification problem and theoretically studies the return–volatility relations. Section 3 presents the Bayesian inference with MCMC methods. Section 4 implements model estimation using S&P 500 index data and discusses the empirical results on the return–volatility relations. Section 5 concludes the paper.

2. Modeling market volatility

2.1. Stock price and volatility dynamics

Under a given probability space (Ω, F, P) and the complete filtration {Ft}t≥0, the market index St is proposed to have the following dynamics

\[ S_t = S_0 \exp \left\{ \left( r + \pi_d T_t^d \right) T_t^d + \left[ W_t^d - k_d(1) T_t^d \right] \right\} \]

\[ + \left\{ X_t^d - k_d(1) T_t^d \right\} \]

(1)

where r is the deterministic risk-free interest rate, \( W_t \) is a Brownian motion, \( X_t \) is a jump component, \( \pi_d \) and \( \pi_x \) are the diffusion and jump risk-premium rates such that the equity risk premium is defined as \( \pi = \pi_d W_t^d + \pi_x X_t^d \), and \( k_d(1) \) and \( k_d(1) \) are convexity adjustments for the Brownian motion and the jump process and can be computed from their cumulant exponents: \( k(u) = \frac{1}{2} \ln[E(e^{iuX})] \), where \( L_0 \) is either \( W_t \) or \( X_t \).

The market index jumps due to the jump component \( X_t \) in (1). Recent studies by Ait-Sahalia and Jacod (2009) and Cont and Mancini (2008) find supportive evidence of infinite activity jumps in stock prices. Therefore, in this paper I apply the Normal Inverse Gaussian process (Barndorff-Nielson, 1998) to model the jump component. The Normal Inverse Gaussian process is an infinite activity pure jump process and can be constructed through subordinating a Brownian motion with drift using an independent subordinator

\[ X_t = \alpha S_t + \eta \tilde{W}(S_t) \]

(2)

where \( \tilde{W} \) is a standard Brownian motion and \( S_t \) is a subordinator that is the inverse Gaussian process \( S_t = IG(t; 1, \nu) \) with the probability density

\[ f_{IG}(x) = \frac{1}{\sqrt{2\pi \nu}} x^{-1} e^{-\frac{x^2}{2\nu}} \]

(4)

The subordinator \( S_t \) is constructed to have unit mean rate and variance rate \( \nu \).

1. With the Brownian subordination \( \tilde{W} \) and the subordinator \( S_t \), the characteristic function of the Normal Inverse Gaussian process can be derived as follows:

\[ \phi(x) = E[e^{ix\tilde{W}}] = \exp \left\{ -\frac{x^2}{2} \left( \sqrt{1 - 2iux} - iux^2 \right) - \frac{1}{2} \right\} \]

(3)

from which its mean, variance, skewness and excess kurtosis can be obtained easily: \( \mu = -\nu \), \( \sigma^2 = \nu \), \( \gamma = -\nu \), \( \text{ESk} = -\nu \), and \( \text{EKurt} = -\nu \).
$T^0$ is a stochastic business time (Clark, 1973; Carr et al., 2003), capturing the randomness of the diffusion variance ($i = 1$) or of the jump intensity ($i = 2$) over the time interval $[0, t]$

$$T^{(i)}_t = \int_0^t V^{(i)}_u \, du,$$  

(5)

which is finite almost surely. $V^{(i)}$, which should be nonnegative, is the instantaneous variance rate ($i = 1$) or the jump arrival rate ($i = 2$), both of them reflecting the intensity of economic activity and information flow. Stochastic diffusion volatility or stochastic jump intensity is generated by replacing calendar time $t$ with business time $T^{(i)}_t$ in the Brownian motion or in the jump process of (1).

The instantaneous variance rate and jump arrival rate are modeled with the square-root processes of Cox et al. (1985)

$$dV^{(1)}_t = \kappa_1(\theta_1 - V^{(1)}_t)dt + \sigma_1 \sqrt{V^{(1)}_t} \, dZ^{(1)}_t,$$

(6)

$$dV^{(2)}_t = \kappa_2(1 - V^{(2)}_t)dt + \sigma_2 \sqrt{V^{(2)}_t} \, dZ^{(2)}_t,$$

(7)

where $Z^{(1)}_t$’s are two standard Brownian motions, which are independent each other, $Z^{(1)}_t$ is correlated to $W_t$ with a correlation parameter $\rho$ in order to accommodate the diffusion leverage effect and is independent of $X_t$, $Z^{(2)}_t$ is independent of both $W_t$ and $X_t$. The long-run mean of $V^{(1)}_t$ is normalized to one since the jump component $X_t$ has non-unit variance.

The above model takes into account the stochastic diffusion volatility and the jump with the stochastic arrival rate. The stochastic jump arrival rate process (7) makes jumps serially correlated, and the model introduces jump clustering. This is consistent to the empirical observation that jumps are indeed autocorrelated and the market crashes are actually realized in a series of jumps over time. Furthermore, even under the lack of the correlation between the index return and the diffusion volatility, the model can generate the time-varying/stochastic skewness and kurtosis through the time-changed jump process.

I explicitly introduce the correlation between Brownian motions $W_t$ and $Z_t$, which accommodates the so-called leverage effect from the diffusion part. The leverage effect of the jump is actually inherent in the above time-changed model because during a time of the high jump arrival rate, the business time flows faster and price jumps occur at an increased rate. If the parameters $\rho$ and $\omega$ are negative, both the diffusion and jump volatility react more to negative returns than to positive returns. In addition to leverage effects, the model also implies the volatility feedback effect. This can be intuitively seen from the diffusion and jump risk premium parts in the asset price process (1). If these risks are priced by the market, the parameters $\pi_W$ and $\pi_X$ should be positive. The market requires a higher return for the riskier asset.

The conditional market return variance $V_t$ has two components: one is from the diffusion risk and the other from the jump risk

$$V_t = V^{(1)}_t + \text{Var}[X_t]V^{(2)}_t,$$

(8)

where $\text{Var}[X_t] = \omega^2 + \eta^2$ is variance of the jump component at time $t = 1$. We can see from (8) that at market crash, the large jump arrival rate $V^{(2)}_t$ can contribute to the abrupt move of market volatility. These two volatility components are fundamentally different. $V^{(1)}_t$ is usually persistent and represents a long-run volatility factor, whereas $V^{(2)}_t$ represents a short-run factor whose effect dissipates very quickly. Therefore, they may have different market prices and demand different compensations. In the following, I call $V^{(1)}_t$ and $V^{(2)}_t$ the diffusion volatility and the jump volatility, respectively. If we set $V^{(2)}_t = 1$ and $\sigma_2 = 0$, we obtain the frequently used constant jump intensity model, which implies i.i.d jumps.

2.2. Risk premia, leverage effects and return-risk relations

To further investigate the interdependence between the return and volatility components, we rewrite our model as follows. For the time-changed Brownian motion, we have

$$W^{(d)}_t \equiv \int_0^t \sqrt{V^{(d)}_t} \, dW_t,$$

(9)

where $d$ indicates that the equality holds in distribution. With this property and the fact $[dW_t, dZ^{(i)}_t] = \rho dt$, the asset price process (1) and the variance rate process (6) can be reformulated into the following form

$$\ln S_t = \ln S_0 + rt + \left(\pi_W - \frac{1}{2}\sigma_1^2\right)T^{(1)}_t + \left(\pi_X - k_X(1)\right)T^{(2)}_t$$

$$+ \int_t^0 \sqrt{V^{(1)}_t} \left(\sqrt{1 - \rho^2}dW^{(d)}_t + \rho dZ^{(1)}_t\right) + X_{r^{(2)}_t},$$

(10)

$$\sqrt{V^{(1)}_t}dZ^{(1)}_t = \frac{1}{\sigma_1} \left[ dV^{(1)}_t - \kappa_1(\theta_1 - V^{(1)}_t)dt \right],$$

(11)

where $W^{(d)}_t$ and $Z^{(1)}_t$ are independent. Note that the Brownian motion $W^{(d)}_t$ in (10) is now different from that in (1). Putting (11) into (10), we obtain

$$\ln S_t = \ln S_0 + rt + \left(\pi_W - \frac{1}{2}\sigma_1^2\right)\int_0^t V^{(1)}_u \, du$$

$$+ \frac{\sigma_1^2}{\sigma_1^2} \int_0^t \left[ dV^{(1)}_u - \kappa_1(\theta_1 - V^{(1)}_u)du\right]$$

$$+ \sqrt{1 - \rho^2} \int_0^t \sqrt{V^{(1)}_u} \, dW^{(d)}_u + (\pi_X - k_X(1)) \int_0^t V^{(2)}_u \, du + X_{r^{(2)}_t},$$

(12)

from which we can derive the relationship between the excess return and volatility components within a time interval $[t, \tau]$, as follows:

$$R_{t, \tau} - rt = -\frac{\kappa_1(\theta_1)}{\sigma_1^2} + \frac{\sigma_1^2}{\sigma_1^2} \int_{t-\tau}^{t} V^{(1)}_u \, du + (\pi_X - k_X(1)) \int_{t-\tau}^{t} V^{(2)}_u \, du + \eta j^{(\eta)}_t \, dW^{(d)}_{\tau^{(d)}_t},$$

(13)

where the Lévy process $X_{\tau^{(d)}_t}$ is substituted with its Brownian subordination form, and the error term $\nu_t = \sqrt{1 - \rho^2} \int_{t-\tau}^{t} dW^{(d)}_{\tau^{(d)}_t} + \eta j^{(\eta)}_t \, dW^{(d)}_{\tau^{(d)}_t}$ has zero mean.

Proposition. Under the model specification (13) and the standard restrictions on parameters $\rho$ and $\omega$ ($\rho < 0$, $\omega < 0$), the expected excess return has the following relationships with its conditional volatility components:

• **Case I:** without leverage effects $\rho = 0$, $\omega = 0$

$$R_{t, \tau} - rt = \left(\pi_W - \frac{1}{2}\sigma_1^2\right)\int_{t-\tau}^{t} V^{(1)}_u \, du + (\pi_X - k_X(1)) \int_{t-\tau}^{t} V^{(2)}_u \, du + \nu_t,$$

(14)

• **Case II:** with leverage effects $\rho < 0$, $\omega < 0$

$$R_{t, \tau} - rt = -\frac{\kappa_1(\theta_1)}{\sigma_1^2} + \frac{\sigma_1^2}{\sigma_1^2} \int_{t-\tau}^{t} V^{(1)}_u \, du$$

$$+ \left(\pi_W - \frac{1}{2}\sigma_1^2\right)\int_{t-\tau}^{t} V^{(1)}_u \, du + (\pi_X - k_X(1)) \int_{t-\tau}^{t} V^{(2)}_u \, du + \nu_t,$$

(15)

where $V^{(1)}_t$ is the unexpected diffusion volatility.
The result of Case I can be obtained straightforwardly from (13) by setting $\rho = 0$ and $\alpha = 0$ and using the Euler approximation to the integrals. Whenever $\rho < 0$ and $\alpha < 0$, using the fact $E_{t-1}[s_{t-1}^{2}] = t/V_{t-1}^{(2)}$, and approximating the unexpected diffusion volatility $V_{t-1}^{(2)}$ with $V_{t-1}^{(1)} = V_{t-1}^{(1)} - V_{t-1}^{(1)}$, which is implied by $E_{t-1}[V_{t}^{(1)}] = V_{t-1}^{(1)} - V_{t-1}^{(1)}$ for a small time interval $\tau$, we have the result of Case II. □

The Proposition shows that in the absence of leverage effects as in Case I, the constant term is zero, and the expected excess return is positively related to both the diffusion volatility and the jump volatility if the risks are really priced by the market. This is in fact Merton’s ICAPM. However, in the presence of leverage effects as in Case II, the constant term is non-zero and should be positive, and the relationships between the expected excess return and the conditional diffusion and jump volatility components are determined by interactions between risk premia and leverage effects. Furthermore, there exists the lead-lag phenomenon, and the unexpected diffusion volatility term $V_{t-1}^{(1)}$ may also play important role in determining the return–diffusion volatility relation. This is actually an extended variant of French et al. (1987) indirect regression except that we have a jump volatility term. Given $\rho < 0$ and $\sigma_{1} > 0$, the excess return has a negative relation with the unexpected diffusion volatility, which provides an indirect evidence of the positive relationship between the expected excess return and the predicted diffusion volatility.

The interdependence between the return and the current and lagged volatility in this paper differs from most of previous studies that mainly focus on one relationship. Whitelaw (1994), Brant and Kang (2004), and Christensen and Nielsen (2007) also document the lead–lag phenomenon. By introducing volatility components and leverage effects, the model explains why empirical investigations have obtained mixed findings because volatility components may behave differently and risk premia and leverage effects play opposite roles in determining relationships between the expected excess return and volatility components. Bollerslev and Zhou (2006) also find that existence of the leverage effect makes it difficult to reveal the positive relationship between the excess return and volatility under the Heston stochastic volatility model.

3. Econometric methodology

This section proposes an efficient Bayesian method to estimate the model. Bayesian estimation with MCMC methods is particularly suitable to the continuous-time financial models (Johannes and Polson, 2009). It simultaneously estimates model parameters and latent states through computing their posterior distributions, delivers exact finite sample inferences, and reduces the computational cost dramatically. Instead of using the traditional Metropolis methods, I propose to apply the more efficient slice sampling method recently developed by Neal (2003). Slice sampler is an adaptive algorithm and is more convenient to draw samples from posterior distributions of latent states.

We have noted from the previous section that for the time-changed Brownian motion, we have the distributional equivalence (9), and for the time-changed Lévy process, we have the Brownian subordination property

$$X_{t}^{\psi} = \omega S_{t}^{\psi} + \eta W(S_{t}^{\psi}),$$

(16)

$$S_{t}^{\psi} = S(T_{t}^{\psi}, 1, v).$$

(17)

With these features, we can transform the model into a state-space representation. Since using the stock price data alone, we have an identification problem for estimating the risk-premium parameters. I thus merge them with the drift and convexity adjustment terms and estimate one parameter $\mu$. Having defined log stock price as $y_{t} = \ln S_{t}$ and discretized the model with a small time interval $\tau$, I obtain the following state-space representation

$$y_{t} = y_{t-1} + \mu \tau + X_{t} + \sqrt{V_{t-1}^{(1)} \tau} 

(18)

V_{t}^{(1)} = V_{t-1}^{(1)} = \kappa_{1} \left(1 - V_{t-1}^{(1)}\right) + \sigma_{1} \sqrt{\tau V_{t-1}^{(1)}},$$

(19)

$$V_{t}^{(2)} = V_{t-1}^{(2)} + \kappa_{2} \left(1 - V_{t-1}^{(2)}\right) + \sigma_{2} \sqrt{\tau V_{t-1}^{(2)}},$$

(20)

$$X_{t} = \alpha S_{t} + \eta \sqrt{S_{t}} \tau,$$

(21)

$$S_{t} = S(T_{t}^{(2)}; 1, v),$$

(22)

where the diffusion volatility $V_{t}^{(1)}$, the jump volatility $V_{t}^{(2)}$, the jump size $X_{t}$, and the subordinator $S_{t}$ are unobservable and regarded as states $H = \{V_{t}^{(1)}, V_{t}^{(2)}, X_{t}, S_{t}\}$, and $\Theta = \{\mu, \omega, \eta, \kappa_{1}, \theta_{1}, \sigma_{1}, \rho, \kappa_{2}, \sigma_{2}\}$ is a set of static parameters that need to be estimated. Eraker et al. (2003) and Li et al. (2008) show that for a small time interval $\tau$, such as daily and/or higher frequency, Euler discretization of the continuous-time models does not introduce significant bias in estimation.

Bayesian estimation tries to find the posterior distribution of parameters and states given the whole set of observations, that is, $p(\Theta, H|Y)$ where $Y = \{y_{t}\}_{t=0}$. With Bayes’ rule, this posterior can be expressed by the likelihood and the prior

$$p(\Theta, H|Y) \propto p(Y|H, \Theta)p(H|\Theta)p(\Theta),$$

(23)

where $p(Y|H, \Theta)$ is the likelihood given states and parameters, $p(H|\Theta)$ is the probability distribution of states conditional on parameters, and $p(\Theta)$ is the prior of parameters. In most cases, direct sampling from the posterior distribution $p(\Theta|H, Y)$ is impossible of its high dimension and complicated form. We can then iteratively draw from its full conditionals $p(\Theta|H, Y)$ and $p(H|\Theta)$ using the Gibbs sampling method. The parameter set $\Theta$ and the state set $H$ can further be broken into smaller blocks.

Since the diffusion volatility and return and returns are correlated, for Bayesian estimation of the correlation parameter $\rho$ and the volatility of volatility parameter $\sigma_{1}$, I follow Jacquier et al. (2004) and reparameterize $(\rho, \sigma_{1})$ to $(z, h)$ with $z = \rho \sigma_{1}$ and $h = (1 - \rho^{2})\sigma_{1}^{2}$. It is not difficult to derive that all the parameters have conjugate priors, that is, given that $\mu$, $\kappa_{1}$, $\theta_{1}$, $\kappa_{2}$, $\omega$ and $z$ have normal prior distributions and $h$, $\sigma_{2}^{2}$, $\eta^{2}$ and $v$ have inverse Gamma prior distributions, their posteriors have the same distributions as those of the priors. The standard methods can be used to sample from these posterior distributions.

For Bayesian estimation of latent states, I follow the single-move approach. We can derive that the jump size $X_{t}$ has a normal posterior distribution (Li et al., 2008). I find that the posterior distribution of $S_{t}$ has a generalized inverse Gaussian distribution $GIG(\alpha, \psi, \chi)$

$$p(S_{t}|Y, H, \Theta) \propto S_{t}^{-2} \times \exp \left\{ -\frac{1}{2} \left[ \left( 1 - \frac{\alpha^{2}}{\psi} \right) S_{t} + \frac{\left( \tau V_{t-1}^{(1)} \right)^{2}}{\psi} + \frac{\chi^{2}}{\psi} \right] \right\},$$

(24)

where $z = -1, \psi = \frac{1}{2} + \frac{\chi^{2}}{\psi}$ and $\chi = \frac{\left( \tau V_{t-1}^{(1)} \right)^{2} + \chi^{2}}{\psi}$. The volatility states $V_{t}^{(1)}$ and $V_{t}^{(2)}$ have non-standard posterior distributions. To sample from these non-standard distributions, MCMC methods have to be applied.

In this paper, I mainly use the slice sampling method recently developed by Neal (2003) when sampling from the non-standard distributions. The method is based on the observation that to sample a random variable, one can sample uniformly from the region under the curve of its probability density function. A Markov chain that converges to this uniform distribution can be constructed by alternately sampling uniformly from the vertical interval defined by the density at the current point and from the union of intervals.
that constitutes the horizontal slices. Eraker et al. (2003) use the usual Metropolis–Hastings algorithm, and Li et al. (2008) propose to use the Adaptive Rejection Metropolis Sampling (ARMS) algorithm. But these methods either need much hand-tuning work or are very computationally intensive, especially when sampling from posterior distributions of latent states. Slice sampling method can adaptively change the scale in choosing slice, which makes it easier to tune than the Metropolis–Hastings algorithm and other methods and avoids problems arising when the appropriate scale of changes varies over time (Neal, 2003). This adaptive property is particularly suitable to draw samples for state estimation.

Slice sampler works with the following steps for a given posterior distribution \( p(x) \):

1. **Step 1:** Starting from an initial value \( x_0 \), uniformly draw a real value \( y \) from \( (0, p(x_0)) \);
2. **Step 2:** Find an interval around \( x_0 \), that contains all, or most of the slices \( S = \{x : y < p(x)\} \);
3. **Step 3:** Uniformly draw a new point \( x \) from the part of the slice within this interval as a sample from the distribution \( p(x) \);
4. **Step 4:** Take \( x \) as a new starting value, and repeat Steps 1–3.

### 4. Estimation results and discussions

This section presents empirical results and implications of the return–volatility relations. Two models are estimated with Bayesian MCMC methods discussed in Section 3. They are the volatility component model (NIG2SV) of Section 2 and one of its nested model: the constant jump intensity model (NIGSV). Sub Section 4.1 presents data. Sub Section 4.2 implements model performance analysis. Sub Section 4.3 discusses the dynamics of volatility components. And sub Section 4.4 investigates empirical evidence of the return–volatility relations.

#### 4.1. Data

The data used to estimate market volatility are S&P 500 stock index ranging from January 4, 1960 to September 30, 2009 in daily frequency, in total 12,522 observations. They are downloaded from Datastream. This dataset is long enough (nearly half century) to contain typical market behaviors we can observe: the recent financial crisis in the late 2008, the market crash on October 19, 1987 (−22.9%), the volatile market and relatively tranquil periods.

Fig. 1 plots the time-series evolution of the (log) index and index returns, and Table 1 presents the descriptive statistics of index returns.

The annualized mean of index returns in this period is around 5.8% and the historical volatility is about 16.1%. A striking feature of data is high non-normality of the return distribution with the skewness \( \frac{0.09}{1.09} \) and the kurtosis 33.0. The Jarque–Bera test easily rejects the null hypothesis of normality of returns with a very small \( p \)-value (less than 0.001). The index returns display very weak autocorrelation.

#### 4.2. Model performance analysis

A couple of questions naturally arise: in modeling S&P 500 index returns and uncovering return–volatility relations, whether is it necessary to introduce the stochastic jump arrival rate? and which model is more suitable?

I first informally check return residuals that are defined as

\[
\epsilon_t = \frac{Y_t - Y_t - \mu^\tau - X_{t+1}}{\sqrt{V_t^{1/\tau}}} 
\]

for each model. Intuitively, if a model is correctly specified, the residuals \( \epsilon_t \) should be asymptotically normally distributed with...
Table 1
Summary statistics of the S&P index returns.

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<th>Mean</th>
<th>Std.</th>
<th>Skewness</th>
<th>Kurtosis</th>
<th>Min</th>
<th>Max</th>
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<tbody>
<tr>
<td>Returns</td>
<td>0.058</td>
<td>0.161</td>
<td>-1.087</td>
<td>32.96</td>
<td>-0.229</td>
<td>0.110</td>
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<tbody>
<tr>
<td>ACF</td>
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<td>-0.046</td>
<td>0.010</td>
<td>-0.013</td>
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</tbody>
</table>

Note: The table presents the descriptive statistics of data for model estimation and empirical analysis. Data are from January 4, 1960 to September 30, 2009 in daily frequency. In total, there are 12,522 observations. Mean and standard deviation are annualized. $\rho$'s stand for the autocorrelations.

Table 2
Bayesian parameter estimates and model comparison.

<table>
<thead>
<tr>
<th></th>
<th>$\kappa$</th>
<th>$\theta$</th>
<th>$\sigma$</th>
<th>$\rho$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIGSV</td>
<td>2.245</td>
<td>0.021</td>
<td>0.235</td>
<td>-0.608</td>
</tr>
<tr>
<td>NIG2SV</td>
<td>(0.322)</td>
<td>(0.002)</td>
<td>(0.013)</td>
<td>(0.030)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$\omega$</th>
<th>$\eta$</th>
<th>$v$</th>
<th>$\mu^*$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIGSV</td>
<td>-0.033</td>
<td>0.073</td>
<td>3.731</td>
<td>0.082</td>
</tr>
<tr>
<td>NIG2SV</td>
<td>(0.013)</td>
<td>(0.006)</td>
<td>(0.465)</td>
<td>(0.015)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>Mean</th>
<th>Std.</th>
<th>Skewness</th>
<th>Kurtosis</th>
</tr>
</thead>
<tbody>
<tr>
<td>Returns</td>
<td>0.004</td>
<td>0.906</td>
<td>-0.019</td>
<td>2.632</td>
</tr>
<tr>
<td>NIG2SV</td>
<td>0.002</td>
<td>0.938</td>
<td>-0.011</td>
<td>2.794</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>$p_D$</th>
<th>$p_0$</th>
<th>DIC</th>
</tr>
</thead>
<tbody>
<tr>
<td>NIGSV</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
<tr>
<td>NIG2SV</td>
<td>–</td>
<td>–</td>
<td>–</td>
</tr>
</tbody>
</table>

Note: Models are estimated with Bayesian methods discussed in Section 3. Posterior means and standard errors (in brackets) of parameters are presented in the first two panels. Panel C presents the first four moments of return residuals computed from Eq. (25), and Panel D reports statistics of the Deviance Information Criterion.

zero mean and unit variance. Table 2 presents the parameter estimates and moments of return residuals computed using these parameter estimates. We can see from the panel C that the NIG2SV model performs better than the NIGSV model, indicating that the stochastic jump arrival rate is necessary.

To formally conduct model comparison, Bayes factor is desirable. It is based on computation of the evidence, the probability of getting data given the model. However, it is not easy to obtain an accurate estimation of the evidence using MCMC sampling. Instead of computing the evidence, Spiegelhalter et al. (2002) propose using the Deviance Information Criterion (DIC), which tackles issues of the goodness-of-fit and the model complexity using an approximate decision-theoretic justification. Indeed, DIC can be shown to be equivalent to the evidence when the deviance is Gaussian. It is particularly convenient for the complex hierarchical state-space models. A recent paper by Berg et al. (2004) has illustrated the potential advantage of this information criterion in determining the appropriate stochastic volatility models.

The last panel of Table 2 presents criteria of the goodness-of-fit ($p_D$), the model complexity ($p_0$), and the Deviance Information Criterion (DIC) for two models. We find that in fitting data, the NIG2SV model is superior to the NIGSV model, but it also has a larger value of the model complexity. When taking into account both goodness-of-fit and model parsimony, we see that the value of DIC for the NIG2SV model is smaller than that for the NIGSV model, indicating the NIG2SV model performs better than the NIGSV model.

4.3. Dynamics of volatility components

We note that both from the distribution of return residuals and the Deviance Information Criterion, the volatility component model (NIG2SV) is preferable. Therefore, in the sequel I focus on the NIG2SV model.

The first two panels of Table 2 presents Bayesian parameter estimates and their standard deviations. Since we use the large amount of data, all parameter estimates are statistically significant.2 The negative estimate of the jump drift parameter $\omega$ ($-0.03$) indicates that the return distribution is left-skewed and negative jumps happen more frequently than positive ones. The relatively large estimate of the jump structure parameter $\nu$ (3.42) shows that the Normal Inverse Gaussian process generates many tiny jumps occasionally accompanied by the large jumps. The estimate of the jump parameter $\eta$ is small with a value 0.08. The high significance of all jump parameters indicates the existence of the jump risk.

The estimate of $\kappa_1$ (2.23) shows that the diffusion volatility $V_1^{(1)}$ is persistent. The volatility of volatility parameter $\sigma_1$ has a small value (0.23), indicating that $V_1^{(1)}$ cannot have a large change at a small time horizon. These two estimates imply that the diffusion volatility affects the return variance in the long horizon. The long-run mean estimate of the diffusion volatility ($\sqrt{\sigma_1}$) is around 14.5% and the correlation estimate is about $-0.61$, which captures the leverage effect and also contribute to the left-skewed distribution of returns.

---

2 In implementation of MCMC simulations, in total 100,000 iterations are run. I collect the result every 10 iterations in order to alleviate the autocorrelation of random variances. As a result, we have 10,000 simulations, among which the last 8,000 simulations are kept for inference. Since the dataset is very large, the likelihoods dominate the priors, which do not make difference in estimation.
In contrast, the jump arrival rate $V_{2t}$ has a very huge mean-reverting parameter ($\kappa_2 = 101$), making the impact of the jump component lasting only in a very short horizon. The jump arrival rate shows very high instantaneous volatility of volatility ($\sigma_2 = 11.6$). These two estimates are highly statistically significant. They indicate that the jump arrival rate can have a sudden increase at a short time interval and can push the return volatility to move up abruptly during market crash.

According to our model specification (NIG2SV), the index return variance is governed by two random sources: $W_t$ and $X_t$, whose rates $V_{1t}$ and $V_{2t}$ are both stochastic and are governed by separate square-root processes. The dynamics of these state variables determine how shocks to $W_t$ and $X_t$ dissipate across the return variance term structure. A transient shock mainly affects the short-run return variance, whereas a persistent shock affects return variance at both short and long horizons. We note that $V_{1t}$ is very persistent. Thus, shocks on $W_t$ shall have long-lasting impacts on the return variance. A recent paper by Christensen and Nielsen (2007) shows that this persistence is caused by presence of long-memory in volatility, which needs to be taken into account when modeling market volatility and investigating the return–risk trade-off. Therefore, I investigate the return–volatility relations using the volatility estimates obtained from the previous subsection. Table 3 presents the summary statistics of the estimated volatility components. The diffusion volatility is highly persistent. The first six autocorrelations are all larger than 0.98. However, the autocorrelation of the jump volatility dissipates very quickly. Its sixth autocorrelation is only about 0.19.

### Table 3

Summary statistics of the estimated volatility.

<table>
<thead>
<tr>
<th></th>
<th>$V_{1t}$</th>
<th>$V_{2t}$</th>
<th>$\sqrt{V_{1t}}$</th>
<th>$\sqrt{V_{2t}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>0.021</td>
<td>1.252</td>
<td>0.130</td>
<td>1.119</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.024</td>
<td>0.062</td>
<td>0.060</td>
<td>0.026</td>
</tr>
<tr>
<td>Skewness</td>
<td>4.481</td>
<td>7.096</td>
<td>1.836</td>
<td>6.099</td>
</tr>
<tr>
<td>Kurtosis</td>
<td>31.60</td>
<td>87.45</td>
<td>9.007</td>
<td>67.76</td>
</tr>
<tr>
<td>Minimum</td>
<td>0.001</td>
<td>1.107</td>
<td>0.027</td>
<td>1.052</td>
</tr>
<tr>
<td>Maximum</td>
<td>0.247</td>
<td>2.357</td>
<td>0.497</td>
<td>1.535</td>
</tr>
<tr>
<td>$\rho_1$</td>
<td>0.998</td>
<td>0.886</td>
<td>0.997</td>
<td>0.889</td>
</tr>
<tr>
<td>$\rho_2$</td>
<td>0.956</td>
<td>0.708</td>
<td>0.994</td>
<td>0.713</td>
</tr>
<tr>
<td>$\rho_3$</td>
<td>0.994</td>
<td>0.535</td>
<td>0.990</td>
<td>0.541</td>
</tr>
<tr>
<td>$\rho_4$</td>
<td>0.992</td>
<td>0.392</td>
<td>0.987</td>
<td>0.396</td>
</tr>
<tr>
<td>$\rho_5$</td>
<td>0.989</td>
<td>0.278</td>
<td>0.983</td>
<td>0.281</td>
</tr>
<tr>
<td>$\rho_6$</td>
<td>0.986</td>
<td>0.187</td>
<td>0.979</td>
<td>0.191</td>
</tr>
</tbody>
</table>

Note: The table presents the summary statistics of the estimated diffusion volatility and jump volatility (both in variance and standard deviation versions) ranging from January 4, 1960 to September 30, 2009. $\rho$'s stand for the autocorrelations.
As shown in Table 2, estimates of $\rho$ and $\omega$ are both negative and significant, indicating existence of the diffusion and jump leverage effects. We thus can see from the Proposition of Section 2 that the return–volatility relations hold for Case II. The relationships are determined by interactions between risk premia and leverage effects. I firstly investigate the direct return–volatility relations by ignoring the unexpected diffusion volatility term in (15)

$$R_t^2 = \alpha_0 + \alpha_1V_{t-1}^{2} + \alpha_2V_{t-1}^{2} + \xi_t.$$  \hspace{1cm} (26)

where $R_t^2$ denotes the excess return between the S&P 500 index return and the risk-free rate that is proxied by the yield of the US 3-month treasury bill, downloaded from Datastream, and $\xi_t$ is a noise term. In order to check the robustness, I also estimate the standard deviation version

$$R_t^2 = \alpha_0 + \alpha_1V_{t-1}^{2} + \alpha_2V_{t-1}^{2} + \xi_t.$$  \hspace{1cm} (27)

We note from Case II that the expected excess return should have a negative relation with the unexpected diffusion volatility. I also investigate this indirect evidence and estimate the following two regressions

$$R_t^2 = \alpha_0 + \alpha_1V_{t}^{(1)} + \alpha_2V_{t}^{(2)} + \xi_t,$$  \hspace{1cm} (28)

$$R_t^2 = \alpha_0 + \alpha_1V_{t}^{(1)} + \alpha_2V_{t}^{(2)} + \xi_t,$$  \hspace{1cm} (29)

where $V_{t}^{(1)} = V_{t}^{(1)} - V_{t-1}^{(1)}$, and others are the same as the above. For a small time interval, $V_{t}^{(1)}$ can be approximately regarded as the unexpected diffusion volatility.

Table 4 presents the parameter estimates and the corresponding t-ratios (in absolute values) based on the Newey–West method with 18 lags for the direct regressions (26) and (27). From panel A that uses the whole dataset, we can see that the constant term is significantly positive in both variance and standard deviation regressions. This is consistent to Case II: if the diffusion leverage effect exists ($\rho < 0$), the constant should be positive. The estimate of $\alpha_1$ is negative, but not statistically significant, indicating that the return–diffusion volatility relation is hard to be determined. The estimate of $\alpha_2$ is negative and highly statistically significant, indicating that the excess return has a negative relationship with the jump volatility.

### Table 4

<table>
<thead>
<tr>
<th></th>
<th>$\alpha_0$</th>
<th>$\alpha_1$</th>
<th>$\alpha_2$</th>
<th>$R^2 (%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>A. Whole period: 1960.01–2009.09</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.020</td>
<td>-0.005</td>
<td>-0.016</td>
<td>0.96</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>(4.257)</td>
<td>(0.361)</td>
<td>(4.196)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.042</td>
<td>-0.001</td>
<td>-0.038</td>
<td>0.91</td>
</tr>
<tr>
<td></td>
<td>(4.327)</td>
<td>(0.198)</td>
<td>(4.284)</td>
<td></td>
</tr>
<tr>
<td><strong>B. Sub-period (1): 1960.01–1987.09</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.014</td>
<td>0.009</td>
<td>-0.011</td>
<td>0.60</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>(2.781)</td>
<td>(0.542)</td>
<td>(2.770)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.030</td>
<td>0.002</td>
<td>-0.027</td>
<td>0.61</td>
</tr>
<tr>
<td></td>
<td>(2.890)</td>
<td>(0.612)</td>
<td>(2.884)</td>
<td></td>
</tr>
<tr>
<td><strong>C. Sub-period (2): 1988.01–1999.12</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.019</td>
<td>0.032</td>
<td>-0.015</td>
<td>1.48</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>(3.665)</td>
<td>(1.997)</td>
<td>(3.673)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.039</td>
<td>0.009</td>
<td>-0.035</td>
<td>1.38</td>
</tr>
<tr>
<td></td>
<td>(3.573)</td>
<td>(2.196)</td>
<td>(3.631)</td>
<td></td>
</tr>
<tr>
<td><strong>D. Sub-period (3): 2000.01–2009.09</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.032</td>
<td>-0.006</td>
<td>-0.026</td>
<td>0.58</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>(2.515)</td>
<td>(0.608)</td>
<td>(2.486)</td>
<td></td>
</tr>
<tr>
<td></td>
<td>0.066</td>
<td>-0.003</td>
<td>-0.059</td>
<td>0.56</td>
</tr>
<tr>
<td></td>
<td>(2.615)</td>
<td>(0.641)</td>
<td>(2.577)</td>
<td></td>
</tr>
</tbody>
</table>

Note: Entries report the parameter estimates and absolute values of the t-ratios (in brackets) for regressions (26) and (27) using the S&P excess returns and volatility estimates from the NIG2SV model.

As a robustness check, I also use different sub-period data: pre-1987’s market crash, post-1987’s market crash and before-Millennium, and after-Millennium. From panels B to D in Table 4, we find that the constant term is always significantly positive, and the estimate of $\alpha_2$ is always negative and statistically significant. However, the estimate of $\alpha_1$ changes its sign and significance level with respect to different periods. For example, over the period of the pre-1987’s market crash, the estimate of $\alpha_1$ is positive, but not significant; over the period of the post-1987’s market crash and before-Millennium, it becomes positive and statistically significant; and over the period of after-Millennium, it is again negative and non-significant. $R^2$’s vary from 0.58% to 1.48% for the variance regressions.

I also implement a recursive procedure to investigate the return–volatility relations. Models (26) and (27) are estimated recursively starting from 1970 with the recursive step being one year. Fig. 3 plots the parameter estimates and the absolute values of t-ratios, which are computed using the Newey–West method with 18 lags. The left panels are for the recursive estimation of parameters and the right panels are for the corresponding t-ratios (in absolute values). The solid line is for the variance regression, and the dashed line for the standard deviation regression. We again find that for both variance and standard deviation regressions, the constant term is always positive and highly significant over time, and the return–jump volatility relation is always negative and highly significant over time. However, the relationship between the excess return and the diffusion volatility is hard to be determined. Its sign changes over time, and it is barely statistically significant over time. The changing sign and almost insignificance of $\alpha_1$ points to the importance of the unexpected diffusion volatility as indicated by the Proposition that shows that the unexpected diffusion volatility needs to be taken into account because of the diffusion leverage effect.

Therefore, I investigate the indirect evidence of French et al. (1987) using models (28) and (29). Table 5 presents the parameter estimates and the corresponding t-ratios (in absolute values) for the indirect regressions. Looking at the panel A that uses the whole data, we find that the constant term is positive and statistically significant, and the return–jump volatility relation is significantly negative. These are the same as those in the direct evidence analysis in Table 4. The estimate of the unexpected diffusion volatility coefficient $\alpha_1$ is negative and highly statistically significant, indicating existence of the indirect evidence: existence of the diffusion leverage effect ($\rho < 0$) implies the negative relationship between the excess return and the unexpected diffusion volatility. Again, as a robustness check, I run the indirect regressions for different sub-periods defined previously. We have the same and robust results as shown from the panels B to D in Table 5. We also note that $R^2$’s increase dramatically with comparison to the direct regressions in Table 4. For example, the $R^2$ is 0.96% for the whole dataset in the direct variance regression, whereas it is increased to 88.24% in the indirect variance regression for the same dataset. Similar increase can also be found for all the sub-period datasets and in the standard deviation regressions.

The negative estimate of $\alpha_1$ in Table 5 provides an indirect evidence of the positive relationship between the expected excess return and the conditional diffusion volatility as explained in French et al. (1987). If the current diffusion volatility is larger than the predicted, the predicted diffusion volatility will be revised upward for all future periods because of its property of high persistence. If the excess return is positively related to the conditional diffusion volatility, the discount rate for future cash flows will increase. The higher discount rate reduces both their present value and the current stock price if the cash flows are not affected.

We have a new finding, that is, the excess return is negatively related to the jump volatility, and this relation is very robust across
different datasets. One explanation for this relationship is very similar to that for the diffusion volatility but in the opposite way. If the current jump volatility is larger than the predicted, the predicted jump volatility will be revised not upward but downward for all future periods because of its property of non-persistence and fast mean-reverting. This means that currently if there are big jumps, in the near future the market anticipates a smaller probability that the stock price drastically jumps downward, and this will make the current stock price higher. This also implies that the market does not price the expected jumps.

Relying on a class of parametric asset pricing models, I show that the market return depends on the diffusion volatility change.

![Figure 3](image-url) Recursive estimation and t-ratios.

### Table 5
Indirect evidence of return–volatility relations.

<table>
<thead>
<tr>
<th></th>
<th>( a_0 )</th>
<th>( a_1 )</th>
<th>( a_2 )</th>
<th>( a_3 )</th>
<th>( R^2 (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Whole period: 1960.01–2009.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.015</td>
<td>-6.864</td>
<td>-0.018</td>
<td>-0.011</td>
<td>88.24</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.030</td>
<td>-1.952</td>
<td>-0.007</td>
<td>-0.026</td>
<td>72.56</td>
</tr>
<tr>
<td>B. Sub-period (1): 1960.01–1987.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.008</td>
<td>-6.973</td>
<td>-0.023</td>
<td>-0.006</td>
<td>92.29</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.014</td>
<td>-1.685</td>
<td>-0.007</td>
<td>-0.012</td>
<td>81.38</td>
</tr>
<tr>
<td>Variance</td>
<td>0.018</td>
<td>-7.078</td>
<td>-0.015</td>
<td>-0.014</td>
<td>89.02</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.040</td>
<td>-1.837</td>
<td>-0.002</td>
<td>-0.035</td>
<td>80.45</td>
</tr>
<tr>
<td>D. Sub-period (3): 2000.01–2009.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.016</td>
<td>-6.635</td>
<td>-0.017</td>
<td>-0.013</td>
<td>93.37</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.014</td>
<td>-2.655</td>
<td>-0.009</td>
<td>-0.011</td>
<td>76.10</td>
</tr>
</tbody>
</table>

Note: Entries report the parameter estimates and absolute values of the t-ratios (in brackets) for regressions (28) and (29) using the S&P excess returns and volatility estimates from the NIG2SV model.

### Table 6
Return–volatility relations implied by volatility innovations.

<table>
<thead>
<tr>
<th></th>
<th>( x_0 )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( x_3 )</th>
<th>( R^2 (%) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A. Whole period: 1960.01–2009.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>0.000</td>
<td>-0.447</td>
<td>-0.041</td>
<td></td>
<td>1.85</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.000</td>
<td>-0.165</td>
<td>-0.091</td>
<td></td>
<td>1.84</td>
</tr>
<tr>
<td>B. Sub-period (1): 1960.01–1987.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>-0.000</td>
<td>-1.290</td>
<td>-0.020</td>
<td></td>
<td>3.77</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>-0.000</td>
<td>-0.303</td>
<td>-0.045</td>
<td></td>
<td>3.21</td>
</tr>
<tr>
<td>Variance</td>
<td>0.001</td>
<td>-0.226</td>
<td>-0.040</td>
<td></td>
<td>2.59</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>0.001</td>
<td>-0.054</td>
<td>-0.091</td>
<td></td>
<td>2.26</td>
</tr>
<tr>
<td>D. Sub-period (3): 2000.01–2009.09</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Variance</td>
<td>-0.002</td>
<td>0.382</td>
<td>-0.062</td>
<td></td>
<td>1.13</td>
</tr>
<tr>
<td>Std. dev.</td>
<td>-0.002</td>
<td>0.123</td>
<td>-0.142</td>
<td></td>
<td>0.95</td>
</tr>
</tbody>
</table>

Note: Entries report the parameter estimates and absolute values of the t-ratios (in brackets) for the regression (30) and its standard deviation version using the S&P excess returns and the estimated volatility innovations from the NIG2SV model.
the diffusion volatility level and the jump volatility level. Christensen and Nielsen (2007) argue that the impact of volatility on stock prices should be short-lived in spite of long-memory in volatility. They claim that it is the innovation to volatility that generates the risk-return relation. Therefore, I investigate a similar empirical model using estimates of volatility components

\[ R_t^f = \alpha + 2z_tV_{1,t}^{(1)} + 2z_tV_{1,t}^{(2)} + \varepsilon_t, \tag{30} \]

where \( V_{1,t}^{(1)} = V_{1,t}^{(0)} - V_{1,t}^{(0)} \) is a proxy of the innovation to each volatility component. Since I use daily frequency in this paper, this proxy is reasonable and innocuous. Table 6 presents the parameter estimates and the corresponding t-ratios (in absolute values) based on the Newey–West method with 18 lags for the model (30). Panel A, which uses the whole dataset, shows that the constant term is tiny, the return–diffusion volatility relation is significantly negative, and the return–jump volatility relation is also significantly negative. As a robustness check, I also use data of different sub-periods. As before, I find that the return–jump volatility relation is very robust and significantly negative, whereas the return–diffusion volatility relation changes its sign and significance level with respect to different datasets and is hard to be identified. The standard deviation regressions result in the same relations. \( R^2 \)'s are slightly improved with comparison to Table 4.

5. Concluding remarks

The return–volatility relation is a fundamental issue in asset pricing theory. This paper investigates this relation by taking into account the model specification problem. We model the stock price dynamics by the time-changed Brownian motion and infinite activity Lévy process, which indicates that aggregate market volatility has two components, one from the diffusion risk and the other from the jump risk. These two volatility components play fundamentally different roles in governing market volatility. The model sheds new light on the return–volatility relation. It implies Merton’s ICAPM in the absence of leverage effects, whereas in the presence of leverage effects, the return–volatility relations are determined by interactions between risk premia and leverage effects. The model provides a theoretical explanation for mixed empirical findings in the return–volatility relation.

Empirically, I find a robust negative relationship between the excess return and the jump volatility, whereas the relationship between the excess return and the diffusion volatility is hard to identify notwithstanding that the indirect evidence of the positive relationship exists. Simply regressing excess returns on aggregate market volatility can not find the inherent return–volatility relations since volatility components have fundamentally different behaviors and opposite roles in impacting the future return changes.

We note that the risk-premium parameters are hard to be identified using stock price data alone. Therefore, it is interesting to investigate the return–volatility relations using both stock price and options data. Joint use of information of both markets can result in accurate estimates of the risk-premium parameters and volatility components. Another interesting direction of research is to investigate the cross-sectional pricing of volatility components (Adrian and Rosenberg, 2008) and how they are related to asset pricing anomalies (Arisoy, 2010; Sun and Tong, 2010; Wright and Zhou, 2009).

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